CLASSICAL LABOR VALUES

Properties of Economic Reproduction

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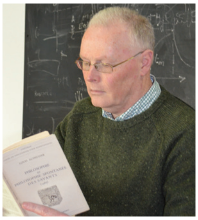


Abstract: We consider economic value as a property that renders heterogeneous goods and services commensurable. Starting from the basic division of output between workers and non-workers, we are able to derive value as a property of systems of economic reproduction. We show its relation to value in classical political economy and in Sraffian economic theory. We go on to discuss the applicability of the derivation and its relevance for analyzing distribution, productivity and employment in a wide range of economies.

Keywords: value; labour value; prices of production

1. Introduction

Economic value enables rational comparisons of economic alternatives by rendering heterogeneous goods and services commensurable. But what is the basis of value? This question repeatedly crops up in practical political discourse (see Bacon and Eltis 1978; Mazzucato 2018).

In the classical approach to political economy, as well as in the early labor movement, value was understood in terms of social labor requirements and was used to study the real distribution of resources, the development of productive capacities, and the accumulation of wealth (see Smith 1776; Ricardo 1817; Marx 1867). In this conception, value is a property of the system of economic reproduction, which is an idea that was elegantly formalized by Sraffa (1960). In Sraffa's analysis it was possible to *derive* value from first principles, rather than just stipulating a definition. For economies with no surplus, the derived values of goods and services did indeed correspond to social labor requirements as in the classical tradition.

For economies with a surplus, however, value could not be derived in Sraffa's framework without additional assumptions. The problem could be overcome by adopting the assumption that rates of return in each sector of production are equalized. While this is certainly consistent with the classical tradition, value would then no longer be proportional to social labor requirements in general. It has been argued that having equal rates of return across all sectors of a capitalist market economy is not only an unstable state of affairs (Farjoun and Machover 1983; Dupertuis and Sinha 2009), it is also contradicted by a substantial body of empirical evidence which shows that profitability varies systematically with capital intensity (see for instance, Cockshott and Cottrell 1998; Zachariah 2006).

In this paper, we raise a different challenge to the Sraffian framework by demonstrating that value can be derived from first principles as a property of surplus-producing economic systems *without* any additional assumptions about rates of return. Starting from a general division of output from such systems, we show that the derived value corresponds to social labor requirements. Moreover, we reconstruct the Sraffian concept of value in the same framework and argue that its applicability belongs to planned economies with constrained forms of investment.

Most economic theory denies valuation is possible outside the relations of commodity-exchanging agents and hence rational comparison of economic alternatives can only exist through market relations.[1] Our conceptualization, however, shows that value can be derived as an endogenous property of any self-reproducing economic system that can redeploy labor across a range of production processes. This includes capitalist market economies, planned economies and mixed state-regulated economies. We proceed to show that this generalized conception addresses central questions that concerned classical political economy and the early labor movement.

2. Economic Reproduction and Value

> The real price of everything, *what everything really costs* to the man who wants to acquire it, is the toil and trouble of acquiring it. (Smith 1776, ch. 5, §2; emphasis added)

We consider an interconnected economic system that is capable of reproducing itself. It produces distinct types of outputs for use,[2] that can be represented by an ordered list of names (such as iron, corn, sugar). Since the list is ordered, we can equivalently represent each output-type by a number,[3] which leads to an efficient representation of d distinct types of outputs, numbered as $1, 2, \ldots, d$. Associated with output-types there are socially defined units of measure: metric tons of steel, bushels of corn, kilograms of sugar, etc. Once the lists of output-types and their units of measure are fixed, they permit representing quantities of heterogeneous products as vectors.

$$\mathbf{b} = \begin{bmatrix} b_{\text{iron}} \\ b_{\text{corn}} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

and

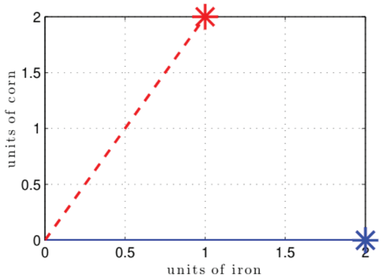
$$\mathbf{b}' = \begin{bmatrix} b'_{\text{iron}} \\ b'_{\text{corn}} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Example 2.1 (Bundles of products): Consider a simple economy with only two output-types: iron and corn. Two different bundles of iron and corn can be expressed as which are visualized in a space of possible bundles in Figure 1(a).

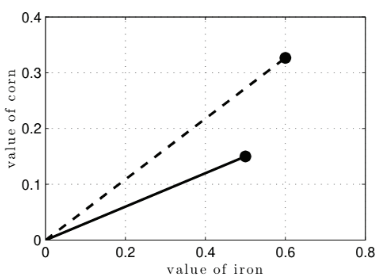
We will proceed as follows. First, we will discuss the form of economic value under consideration. Next, we consider economic reproduction with a workforce, which divides the net product of the system. Finally, we show that, using an invariance condition for the real distribution between workers and non-workers, it is possible to deduce economic value from these general considerations. We will make use of some basic variables, summarized in Table 1.

2.1. Form of Value

The set of product bundles allows operations such as addition, **b**+**b′**, and multiplication by a scalar, 2**b**.[4] The most basic point about *economic value* is that it permits also the ordering of product bundles, so that **b** is less than (or equal to) **b′**.



(a) bundles **b** (solid) and **b′** (dashed)



(b) **v** (solid) and **v*** (dashed)

Figure 1 (a) Bundles of Heterogeneous Products Represented as Vectors. (b) Possible Valuations Represented Using Vectors

Notes: All valuation vectors along a ray are equivalent as they result in the same ordering of bundles.

Table 1 Glossary of Variables Used Below

Variable	*Meaning*
$\mathbf{b}$	product bundle
$\mathbf{n}$	net product bundle of the economy
$\mathbf{v}$	valuation vector
$\boldsymbol{\kappa}$	real-consumption rate vector
$\mathbf{R}$	workforce consumption requirement matrix
$\mathbf{H}$	production requirement matrix
σ	share of surplus-value
u	share of unproductive consumption
g	uniform expansion rate

That is, value orders heterogeneous bundles of products by mapping them onto commensurable units. We formalize the ordering of product bundles using the following definition.

Definition 2.2 (Ordering heterogeneous bundles): Any pair of product bundles, **b** and **b'**, can be ordered with respect to their value, denoted

$$\mathbf{b} \leq_{\mathbf{v}} \mathbf{b}' \iff \mathbf{vb} \leq \mathbf{vb}',$$

where **v** is a valuation (row) vector that specifies non-negative values for each unit of output. Value is thus specified by **v** and enables a preordering of bundles.[5]

Example 2.3 (Valuation): In the simple economy above, value is specified by

$$\mathbf{v} = \left[v_{\text{iron}} \; v_{\text{corn}} \right]$$

which encodes amount of value per unit of iron and corn. Then the value of bundles **b** and **b'** in Example 2.1 equal

$$\mathbf{vb} = 2 \cdot v_{\text{iron}}$$

and

$$\mathbf{vb}' = 1 \cdot v_{\text{iron}} + 2 \cdot v_{\text{corn}},$$

respectively. Using the valuation vector **v** = [0.5 0.15], illustrated in Figure 1(b), the first bundle is more valuable than the second, $\mathbf{b} \leq_{\mathbf{v}} \mathbf{b}'$. Note any two valuation vectors that are proportional to each other yield the same ordering and are thus equivalent valuations. Using an alternative valuation vector **v**, however, results in the opposite order $\mathbf{b}' \leq_{\mathbf{v}} \mathbf{b}$.

Remark 2.4: The use of vectors to describe heterogeneous goods and services formalizes the analysis of economic value discussed in Marx (1867, ch. 1). It was pioneered by physicists who had played a key role in formalizing quantum mechanics using the same tools (von Neumann 1945, 1955; Thompson and Weil 1971). The analysis in Section 2.3 will build on insights from this formalization.

This raises a series of questions. Is value merely stipulated or can it be derived from economic principles? When and why do distinct products have any value? What indeed does it mean to have economic value? Prices may appear to provide a valuation of product bundles, since they assign a quantity of money to each unit of a product. But market prices observed in commodity exchange randomly fluctuate from one transaction to the next and, as the classical economists understood well, the very notion of goods being over- and under-priced implies that value is more fundamental than prices.

In what follows we will show that the formal and quantitative properties of value can be derived from something more fundamental than commodity exchange, namely, the technical and social structure of reproduction. That is, we will *derive* **v** using two basic assumptions:

- Value is a real cost that only changes with the structure of economic system.
- Labour can be trained and redeployed across economic activities.

We will then relate this result to the framework in Sraffa (1960) and draw out some implications for economic analysis.

2.2. Production and Consumption by Workforce

Over a given period, the economic system produces a large bundle of goods and services. After deducting the intermediate inputs consumed in the overall production process, the economy produces a *net product* of goods and services that are consumed, invested, or hoarded by people. We shall denote this net product bundle as **n**.

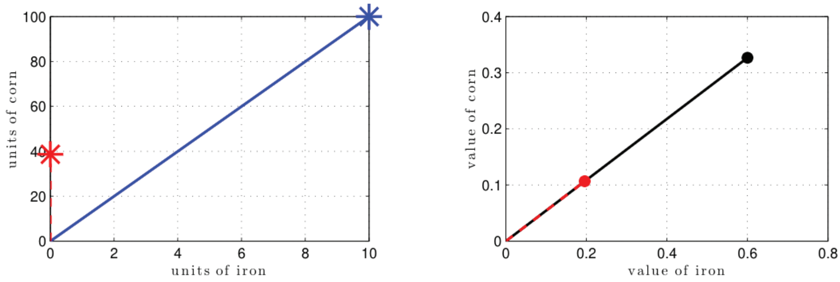
Example 2.5 (Net product): Suppose the net output of the economic system is

$$\mathbf{n} = \begin{bmatrix} n_{\text{iron}} \\ n_{\text{corn}} \end{bmatrix} = \begin{bmatrix} 10 \\ 100 \end{bmatrix},$$

then

$$\mathbf{vn} = 10 \cdot v_{\text{iron}} + 100 \cdot v_{\text{corn}}$$

is the value of the net product, or total value added, yet to be defined. For an illustration of the bundle **n**, see Figure 2(a).



(a) bundles **n** (solid) and **Rn** (dashed)

(b) **v** (solid) and **vR** (dashed)

Figure 2 (a) The Net Product and the Consumption Requirements of the Workforce Determined by **κ**. (b) A Special Valuation Vector **v** that Is Only Rescaled After Transforming it by **R** (see the invariance condition (7))

Economic reproduction requires work so that one part of **n** is necessarily consumed by the workforce and its dependents.

Definition 2.6 (Real-consumption rates): For a given period let **κ** denote the vector of consumption rates, which records the total amount of each output consumed by the workforce divided by the number of units of labor deployed.[6] If the time period is a week and the unit of measurement of labor a working week, **κ** describes the average amount of goods consumed by workers each week.

Example 2.7 (Real-consumption rates): During a given period, suppose the average consumption rate in the simple economy is one unit of corn per person week deployed. Then we can write

$$\boldsymbol{\kappa} = \begin{bmatrix} \kappa_{\text{iron}} \\ \kappa_{\text{corn}} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The consumption rates of the workforce determine the real division of the net product **n** between workers and non-workers.

Result 2.8 (Consumption requirement matrix): Given **κ** and the technical conditions of production, there exists a unique consumption requirement matrix **R**(**κ**) such that **Rn** equals the bundle of goods and services consumed by the workforce.[7]

Example 2.9 (Reproduction of simple economy): Consider the net product **n** in Example 2.5 and the consumption rates **κ** in Example 2.7. Suppose the technical conditions of production are described as follows: one unit of iron requires on average 0.6 units of labour; and one unit of corn requires 0.2 units of labour, 0.2 units of steel and 0.02 units of corn.[8]

Then the necessary consumption of the workforce is the bundle

$$\mathbf{Rn} = \begin{bmatrix} 0 & 0 \\ 0.60 & 0.33 \end{bmatrix} \begin{bmatrix} 10 \\ 100 \end{bmatrix} = \begin{bmatrix} 0 \\ 39 \end{bmatrix} \tag{4}$$

See Figure 2(a) for an illustration of the bundle **Rn**.

After deducting the consumption by the workforce, what remains of the net product is

$$\text{surplus product} = \mathbf{n} - \mathbf{Rn}, \tag{5}$$

consisting of investment goods, luxuries, and so on. Given the consumption rates $\boldsymbol{\kappa}$ of the workforce, *any* net product **n** is divided between workers **Rn** and non-workers **n** – **Rn** in definitive proportions.

2.3. Deriving Economic Value

> All economists share the error of examining *surplus-value* not as such, in its pure form, but in the particular forms of profit and rent. (Marx 1999, ch. 1, §1; emphasis added)

The real division of the net product also distributes its value **vn** between workers and non-workers.

Definition 2.10 (Share of surplus-value): The real distribution of the total value added is quantified by the share of surplus-value,

$$\sigma = \frac{\mathbf{vn} - \mathbf{vRn}}{\mathbf{vn}}, \tag{6}$$

and is bounded between 0% and 100%.[9] It is determined by the consumption rates $\boldsymbol{\kappa}$.

We will now show that value can be *derived* as an endogenous property of economic reproduction. For the consumption rates to determine the distribution of value of *any* net product in a definitive proportion σ, the valuation vector in (6) must satisfy the following invariance condition

$$\boldsymbol{vR} \propto \boldsymbol{v} \tag{7}$$

Then σ is determined by $\boldsymbol{\kappa}$ and is invariant to **n**. Only certain valuations satisfy this condition, since it means that the values of each output-type must be proportional to the values of the consumption requirements needed to reproduce them.

Result 2.11 (Determination of value): The valuation vector **v*** that satisfies the invariance condition (7) is unique up to a unit of choice. The real distribution of value is then determined in definitive proportions by the consumption rates and σ is obtained from the maximum eigenvalue of **R**.[10] Thus value can be derived from first principles as a property of economic reproduction with a workforce.

Example 2.12 (Derived value): The following valuation vector

$$\mathbf{v}^* = \begin{bmatrix} v_{\text{iron}} & v_{\text{corn}} \end{bmatrix} = \begin{bmatrix} 0.60 & 0.33 \end{bmatrix}$$

satisfies the invariance condition (7) since it is proportional to the transformed vector

$$\mathbf{v}^*\mathbf{R} = \begin{bmatrix} 0.20 & 0.11 \end{bmatrix},$$

as seen in Figure 2(b). Then the real distribution of value is determined by the real-consumption rates and invariant to the form of the net product. The valuation vector **v*** is obtained as a left-eigenvector to **R** and we see that a unit of iron is roughly twice as valuable as a unit of corn.

Remark 2.13: The use of invariance conditions, such as (7), in economic analysis was pioneered by von Neumann (1945). The condition (7) is embedded in the analysis of economic reproduction found in Marx (1885, pt. 3).

2.4. Connection to Classical Political Economy

> The value of a commodity [. . .] depends on the relative quantity of labour which is necessary for its production, and *not* on the greater or less compensation which is paid for that labour. (Ricardo 1817, ch. 1, §1; emphasis added)

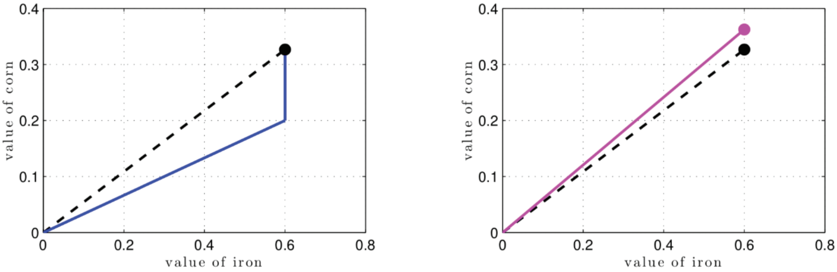
Result 2.14 (Connection to classical economics): The valuation vector **v** that satisfies the invariance condition (7) is obtained by integrating all coexisting labor requirements in production,

$$\mathbf{v}^* \propto \boldsymbol{l}(0) + \boldsymbol{l}(1) + \boldsymbol{l}(2) + \ldots \tag{8}$$

where $\boldsymbol{l}(k)$ is a vector of labor requirements of the kth intermediary inputs in production.[11] Value is therefore invariant to changes in workers' consumption or the distribution of the net product.[12]

Example 2.15 (Integration in simple economy): The coexisting labor requirements $\boldsymbol{l}(0)$, $\boldsymbol{l}(1)$, $\boldsymbol{l}(2)$, . . . for each output-type are illustrated in Figure 3(a). By adding them all, we see that they correspond to **v***.

Remark 2.16: The value of a product bundle **b** is thus obtained by an integration $\left(\boldsymbol{l}(0) + \boldsymbol{l}(1) + \boldsymbol{l}(2) + \ldots\right)\mathbf{b} = \mathbf{v}^*\mathbf{b}$ and corresponds to direct and indirect labor requirements. Contrary to the characterization in Mirowski (1989), classical labor value is not an intrinsic property of products, but is rather an integral property that



(a) $\boldsymbol{\ell}(k)$ (solid) and $\mathbf{v}^*$ (dashed) (b) $\mathbf{v}^*$ (dashed) and $\widetilde{\mathbf{v}}$ (solid)

Figure 3 (a) By Adding All Coexisting Labor Requirements at Step k=0, 1, . . . in Production, We Obtain **v***. (b) Valuation Vectors **v*** and $\tilde{\mathbf{v}}$ that Satisfy Two Different Invariance Conditions

reproducible goods and services acquire from the economic system (see also Wright 2015, ch. 6). In this sense **v*b** is similar to the form of an electrostatic field at a given point in space, which is an integral over space of all point charges whose effect rapidly declines with distance.

Remark 2.17: The sum of nonnegative terms in (8) converges for an economy capable of reproducing itself and the derived value is economically meaningful also when units of production can produce several output-types. This is in contrast to the definition stipulated for highly constrained settings in Steedman (1977, 11–12) (see also Farjoun 1984).

The production of value is therefore inseparable from socially organized production of real goods and services. It is distinct from monetary income generation; rather money and prices are symbolic means by which value is claimed and distributed in market economies. The natural unit of value is worker-time and using such units we shall call **vb** the "labour value" of product bundle **b**.[13]

The specific economic form, in which unpaid surplus-labor is pumped out of direct producers, determines the relationship of rulers and ruled, as it *grows directly out of production itself* and, in turn, reacts upon it as a determining element (Marx 1894, ch. 47; emphasis added).

Result 2.18 (Surplus labor-time): Using **v***, the share of surplus-value σ in (6) equals the fraction of work in the economy employed to reproduce the surplus product.

Example 2.19 (Share of surplus): Consider the real-consumption requirement matrix **R** in (4). Its eigenvalue is readily computed, and yields the following share of surplus-value σ = 67%. Thus 67% of the work in the economy is materialized in the form of surplus outputs.

In economic systems for which the real distribution of value is not controlled by the workforce, their surplus labor is extracted and consumed by a distinct economic class.

3. Relation to Sraffian Value Theory

The preceding results show that value can be derived from first principles as a property of economic reproduction and that it is invariant to the real distribution between workers and non-workers. This invariance is explicitly denied in the analysis of Sraffa (1960), which derives value in a capitalist market economy where $\boldsymbol{\kappa}$ corresponds to the average real-wage rate. Unlike the derivation above, which begins from a general macroeconomic accounting of the net product, Sraffa's derivation starts from accounting the costs of producing commodities in each sector of the economy. To account for surplus production, an additional assumption must be introduced: equalized rates of return across all sectors.

This hypothetical equilibrium state of the economy is often justified by referring to the classical notion of competition. For the state to be stable in formal models requires a delicate balance of price and quantity adjustment mechanisms (see Dupertuis and Sinha 2009; Wright 2015, ch. 7). But in real economies, this state cannot be stable since the competitive forces that tend to scramble rates of return are at least as intrinsic and powerful as those that equalize them (Farjoun and Machover 1983). In fact, real economies exhibit a systematic negative association between rates of return and capital intensity (Cockshott and Cottrell 1998; Zachariah 2006).

Using a basic division of the net product as above, we will now show that Sraffa's theory of value can be derived under conditions of constrained investments, without any economic equalization mechanism. Recall that the surplus product $\mathbf{n} - \mathbf{Rn}$ is a bundle of goods and services for investment and surplus consumption.

Definition 3.1 (Share of unproductive consumption): The share of total value added $\mathbf{vn}$ that is unproductively consumed surplus is denoted

$$0 \leq u \leq \sigma.$$

Result 3.2 (Production requirement matrix): There exists a unique requirement matrix $\mathbf{H}$ such that $\mathbf{Hn}$ equals the bundle of goods required in the production process.[14]

Example 3.3 (Production requirements): The product bundle required to reproduce the net product in Example 2.5, under technical conditions described in Example 2.9, are

$$\mathbf{Hn} = \begin{bmatrix} 0 & 0.20 \\ 0 & 0.02 \end{bmatrix} \begin{bmatrix} 10 \\ 100 \end{bmatrix} = \begin{bmatrix} 20 \\ 2 \end{bmatrix}$$

That is, an average of 20 units of iron and 2 units of corn need to be consumed in the production process.

Definition 3.4 (Constrained investments): Suppose the scale of production inputs is to be expanded by a uniform rate $g \geq 0$. This constrains the bundle of investment goods to be g**Hn**.

Result 3.5 (Unproductive consumption): Under constrained investments, the share of unproductive consumption equals

$$u = \sigma - g\frac{\mathbf{vHn}}{\mathbf{vn}} \geq 0 \tag{10}$$

The minimum, u=0, is attained at the maximum rate $g_{\max}$ that is obtained by the maximum eigenvalue of $\left((\mathbf{I}-\mathbf{R})^{-1}\mathbf{H}\right)$.[15]

When using labor values **v***, the share of surplus-value is determined by **κ** in a definitive proportion σ that is invariant to **n**. Suppose we drop this invariance condition, and instead require the share of unproductive consumption to be determined by g in definitive proportion u that is invariant to **n**.

Result 3.6 (Alternative derivation of value): Under a regime of constrained investments, the valuation vector $\tilde{\mathbf{v}}$ that ensures that the unproductive share in (10) is invariant to **n** is unique up to a unit of choice. It is given as a transformation of labor values,

$$\tilde{\mathbf{v}} = \mathbf{v}^*\left(\mathbf{I} - \frac{g}{1-u(g)}\mathbf{H}\right)^{-1} \tag{11}$$

and the share of unproductive consumption $u(g)$ is obtained from the maximum eigenvalue of $(\mathbf{R} + g\mathbf{H})$, irrespective of **n**. Transformed value is therefore no longer invariant to changes in workers' consumption or the distribution of the net product.[16]

Example 3.7 (Transformed value): Suppose the uniform expansion rate is g=0.10. Then the transformed valuation vector

$$\tilde{\mathbf{v}} = \begin{bmatrix} v_{\text{iron}} & v_{\text{corn}} \end{bmatrix} = \mathbf{v}^*\left(\mathbf{I} - \frac{g}{1-u(g)}\mathbf{H}\right)^{-1} = \begin{bmatrix} 0.60 & 0.33 \end{bmatrix}\begin{bmatrix} 1 & 0.06 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.60 & 0.36 \end{bmatrix}$$

ensures that (10) is invariant to **n**. This valuation differs slightly from the labor values in Example 2.12, as illustrated in Figure 2(b).

Result 3.8 (Equivalence): The transformed values are equivalent to labor values, $\tilde{\mathbf{v}} \propto \mathbf{v}^*$, only if the average capital intensities in the production of each output-type are equal or $g = 0$.[17]

Result 3.9 (Connection to Sraffian economics): The transformed values $\tilde{\mathbf{v}}$ can be decomposed into weighted labor requirements,

$$\tilde{\mathbf{v}} \propto \boldsymbol{l}(0) + (1+\pi)\boldsymbol{l}(1) + (1+\pi)^2 \boldsymbol{l}(2) + \ldots \quad (13)$$

where $\pi = g/(1-u(g))$ from (11). Thus $\tilde{\mathbf{v}}$ is equivalent to Sraffa's natural prices with an average rate of return π.[18]

Starting from the same level of abstraction as Sraffa, his conception of value can therefore be derived without the implausible assumption of equal rates of return in a capitalist economy. Instead, it requires constrained forms of investment, which relegates the application of Sraffian natural prices to the domain of planned economies (see Brody 1970).

By contrast, the derived labor value does not require additional assumptions on equalization or constrained investments. Under this endogenous valuation, the rates of return across sectors decrease with capital intensity, which is indeed an association found in real capitalist economies. It may be asked how this can arise against market arbitrage? The relation appears to be an emergent property arising from the constraint on firms, integrated through economic reproduction, to meet their wage costs (Farjoun and Machover 1983; Cockshott and Cottrell 1998; Zachariah 2006). A formal investigation of the circumstances under which either Sraffian natural prices or labor value may be attractors of dynamical processes of economic reproduction is an open research topic.

4. Applicability of Derivation

We have considered economic value as a property that renders heterogeneous products commensurable. We then derived value as a property of an economic system starting from a basic division of the net product (6) and applying an invariance condition to its distribution. The derived value was shown to be a generalization of the classical conception of value as labor requirements in social production. This, however, raises the question about the economic conditions in which the derivation is applicable.

> Every child knows [. . .] that the masses of products corresponding to the different needs required different and *quantitatively determined masses of the total labor of society*. That this necessity of the distribution of social labor in definite proportions *cannot* possibly be done away with by a particular form of social production but can only change the mode of its appearance, is self-evident. No

> natural laws can be done away with. What can change in historically different circumstances is *only the form* in which these laws assert themselves. (Marx 1868, §1; emphasis added)

In our formalization, the net product **n** represents the "masses" of different products and the corresponding elements of **v*** quantitatively determine the masses of total labor of society required. The key assumption of our derivation is an economically meaningful definition of real-consumption rates **κ** per unit of general labor time, which divides the net product in definitive proportions. If labor inputs across different processes cannot be aggregated to commensurable units of time, then **v*** cannot be derived as a fundamental property in our analysis. For economic systems that are capable of training and redeploying their finite amount of available labor time across different production processes, value is a derivable property that orders heterogeneous products with respect to labor requirements.

It is in the continual process of training and redeployment of laboring capacity across production that an economic system renders concrete work tasks as an expenditure of a commensurable abstract labor resource, quantified in units of time.[19] This enables a rational comparison of economic alternatives and would include a range of self-reproducing economic systems. Did, for instance, value and the disposition of labor time matter to the slave lords of antiquity? According to Cato, it appears that they did:

> When [the master of a farmstead] has learned the condition of the farm, what work has been accomplished and what remains to be done, let him call in his overseer the next day and inquire of him what part of the work has been completed, what has been left undone; whether what has been finished was done betimes, and whether it is possible to complete the rest; and what was the yield of wine, grain, and all other products. Having gone into this, he should make a *calculation of the labourers and the time consumed.* (Hooper and Ash 1935, 9; emphasis added)

In the slave plantations described above by Cato, the disposition of labor is self-evident and "natural," it is not obscured by monetary indirection. But it is still labor in the abstract, albeit of a given group of slaves, being distributed between concrete tasks: meadow clearing, faggot bundling, road-work, etc.

The necessity to take into account the usage of labor time, whether that be the time of slaves, wage laborers, citizens of a socialist commonwealth, is a natural necessity that could not be abolished, only change its historical form. By contrast, in economies with institutions that prevent the redeployment of workers across tasks, e.g., rigid forms of caste hierarchies, there can be no general labor resource quantifiable in commensurable units.

4.1. Capitalist Market Economies

In a capitalist economy, the necessity to distribute labor appears as simply expenditures of money on wages to top-level managers in decentralized firms. So, the wage budget allocated to different branches of a firm provides an indirect representation of the needed allocation of labor.

As one descends the management hierarchy, the simple monetary view of things becomes insufficient. The subsidiary managers have to allocate specific people to specific tasks just as the slave overseer had to. By contrast, as one moves further away from the production process, the representation of labor becomes increasingly obscure and monetary. Indeed, when the products of the economy are allocated between agents as commodities, the monetary calculations are based on market prices which randomly fluctuate from one transaction to the other. The relation between market prices of commodities and their labor values is necessarily a statistical one, and there is a substantial body of theoretical and empirical work establishing this relation.[20] To an individual, money appears to be freely disposable between different products, but in reality, such choices are limited by macroeconomic constraints set by $\mathbf{v}^*$, which represent real costs irrespective of random market prices.

Nevertheless, firms in a capitalist market economy do solve labor allocation problems via decentralized monetary calculations. The feasibility of this monetary accounting mechanism rests on the fact that human labor is flexible and can be redirected, either within the firm or on the employment market, between activities. In capitalism, the redeployment of labor between concrete tasks across the production system occurs through the transfer, hiring, and firing of workers within and across decentralized firms. This allows a single scalar measure like money to function as a system of social accounting.

4.2. Soviet-Type Planned Economies

Planned economies too have to grapple with the finite nature of their labor supplies, and the need to expend effort for any worthwhile effect. This implies that they too will have to have social forms in which this necessity will be expressed. The necessity for the labor force to be allocated in a manner determined by technical conditions took in the planned Soviet-socialist economies the form of a directive plan of $\mathbf{n}$. This plan involved drawing up material and labor balances for the overall economy. We know that Soviet-socialist economies continued to use monetary calculations, which, to a greater or lesser degree of adequacy, allowed indirect calculations to be done on social labor requirements. While monetary calculation and allocation in capitalist market economies redeploy a certain amount of labor via the recreation of a pool of the unemployed, the Soviet-socialist economies did not develop the kind of labor time accounting, planning, and regulation that would be required to carry out reallocations of labor within a fully employed workforce.

4.3. Capitalist War Economies

In capitalist war economies, production, by and large, still took place in privately owned firms. There were state munitions factories like the Royal Arsenal or the Oak Ridge and Los Alamos atomic weapons plants, but these were exceptions. The state directed labor, by conscripting it into the army, and by conscripting women and men in key trades into essential war work. It also rationed the supply of key materials, fuels, and foodstuffs. Firms were subject to negotiated direction to produce only munitions, or restricted ranges of "utility" products (Edgerton 2011). Money was still used to pay for the munitions delivered, and to pay workers. Buying food required both money and ration cards. Money alone was not enough either for the consumer or for firms. In peace, money as the universal ration constrains everything. Shortage of it constrains the working-class consumers and uncertainty about future revenue constrains even those firms which have good cash reserves. Because the constraint on production comes via market exchange in price units rather than directly in units of products, peace-time capitalist market economies typically operate somewhat below full capacity. In war, national survival dictates that every available resource be put to use. The economy operates at the limits of its physical resources in materials, people, and machines.

The state as primary purchaser has to look not just at the projected costs of ships, aircraft, etc., that it is ordering, but at all sorts of material constraints. In deciding what type of destroyers to order, the Navy first took into account the requirements of their admirals for the ships to carry guns of different types, torpedoes, and anti-submarine weapons: all technical not financial issues. They then had to take into account the number of shipyards in the country able to build ships of different sizes, the delivery schedules for different kinds of projected weapons and ship machinery, the availability of metals and alloys of different weights and strengths. They then had to ask whether the demands on skilled labor would require the cancellation or postponement of other orders.[21] Money was a relatively secondary concern. The availability of state credit that, at least within the domestic economy, was effectively unlimited, removed money as a constraining resource (Keynes 2010). The same point about money applied *a fortiori* to the Soviet-socialist economies. Money was never a constraint for them. Labor, plus available plant and equipment, however, were.

4.4. Non-Monetary Planned Economies

If we imagine a planned economy that does away with money altogether, then it would still need value for economic comparisons. In the analysis above we have considered a linear form of value, **vb**. For economies based on market-exchange, alternative value forms would easily render them unviable, since under a set of equal value exchanges, agents would then end up with more goods or services than they started with (see Cockshott 2009; Cockshott and Cottrell 2004).

While the linear value form induces an economically meaningful way to compare heterogeneous product bundles, it remains an open question whether other forms are viable. In the early labor movement, there were ideas of using labor time to allocate consumer goods (e.g., Marx 1970). But value accounting would also be needed to set budgets for public projects. Research programs to develop vaccines, or to explore the moons of Jupiter, would need some limits posed on the amount of social labor that they could use. The same applies to general democratic decisions about long-term structural investment. Society as a whole could not meaningfully decide what portion of its output should be devoted to investment and research, that is to say σ in our notation, unless the surplus was measurable. As we have shown, a consistent measure of this surplus allocation (6) is labor value.

4.5. Fully Automated Economies?

It may be objected that some future society may have at its disposal a class of universal robots, so skilled and dexterous, so intelligent and adaptable, that these beings may come to supplant us in our toils. Is value a property of economic reproduction in this thought experiment?

The robots would need energy, require repair, and absorb the effort of other robots in their initial construction. If the robots are universal and redeployable across concrete tasks, then their reproduction defines real-consumption rates $\boldsymbol{\kappa}$ per unit of general robot labor time. In this case, value follows from our analysis, with this simple proviso, that the labor time is to be understood as redeployable general robot time. Humans, in this hypothetical society, would be in the position of slave-owning ancients: idlers depending on the surplus labor of others.

5. Implications for Labor

We now proceed to apply the generalized conception of value to central questions that concerned classical political economy and the early labor movement: Economic inequality, productivity, employment, and the utilization of surplus economic capacity.

5.1. Total Productivity and Employment

> The values of commodities are directly as the times of labour employed in their production, and are *inversely* as the productive powers of the labour employed. (Marx 1865, sec. IV, §17; emphasis added)

Technical and organizational changes in the economy alter the coexisting labor requirements in production (8).[22] For notational simplicity, let

$$\dot{\mathbf{v}}^* = \frac{d}{dt}\mathbf{v}^*$$

denote the change in labor values per unit of time. This quantity has profound effects on both production and employment.

Result 5.1 (Productivity): Suppose the labor value of output-type i is reduced at the relative rate

$$\rho_i \equiv -\dot{v}_i / v_i$$

Then, for a given level of employment, the net output of i can grow at the relative rate ρ_i. Thus labour values are (inverse) measures of total productivity in the economy.

Example 5.2 (Labor value and total productivity growth): The labor value of corn can be lowered by decreasing the amount of coexisting labor requirements. Thus technical improvements in the production of iron affect the labor value of corn. Suppose its unit value, v_{corn}, decreases by the rate $\rho_{corn} = 5\%$ per annum. Then the capacity to produce corn doubles roughly every 14 years.

Result 5.3 (Employment): Suppose the final demand for output-type i grows at the relative growth rate g_i. Then the total demand for labor changes by the rate $g_i - \rho_i$. Thus labor values are employment multipliers in the economy.[23]

Economies with institutions that progressively lower the labor values of the outputs are capable of rapidly increasing material living standards as well as leisure time. At the same time, economies that lack coordination between technical change and changes in consumption and investment demands can give rise to both persistent unemployment and chronic labor shortages. When $g_i < \rho_i$, the total demand for labor declines exponentially and must be compensated by increased demand among other outputs to prevent the rise of unemployment.

5.2. Basic Outputs and Surplus-Value

The relations between the products of the economy are in general not symmetric: some output-types enter directly or indirectly as inputs to all goods and services, while other output-types do not. Certain distributional consequences can be derived from this asymmetry by extending the analysis in Sraffa (1960) to the theory of surplus-value developed in Marx (1867).

Definition 5.4 (Basic and nonbasic outputs): *Basic* outputs are directly and indirectly required in the production of all outputs, while the *nonbasic* outputs are not.[24]

Example 5.5 (Basics and nonbasics): We consider extending the simple economy to three output-types: iron, corn, and sugar. The technical conditions described in Example 2.9 are extended so that reproducing a unit of sugar requires 0.3 units of labor, 0.3 units of iron, and 0.1 units of corn. Since sugar does not enter the reproduction of all other outputs in this economy, it is a nonbasic output-type.[25]

The production of basic outputs forms a self-reproducing sector of the economy which is critical in determining the share of surplus-value σ.

Result 5.6 (Determinants of surplus share): The share of surplus-value is determined by productivity in and the workers' consumption from the basic sectors of the economy. That is,

$$\sigma = 1 - \mathbf{v}_b^* \boldsymbol{\kappa}_b,$$

where $\mathbf{v}_b^*$ and $\boldsymbol{\kappa}_b$ are the labor values of basic outputs and corresponding worker's consumption rates, respectively. Luxuries and other nonbasic outputs do not affect σ.[26]

Remark 5.7: The share of surplus-value σ increases when the labor values of basic outputs decrease faster than the workers' consumption rates increase. This requires technical change in the basic sector. But σ can also increase by extending the number of working hours without compensation so that the consumption rates $\boldsymbol{\kappa}_b$ fall. These paths correspond to the distinction between "relative" and "absolute" surplus-value described in Marx (1867, pt. 3–5). Note that the nonbasic sector therefore cannot give rise to "relative" surplus-value (see Cockshott and Zachariah 2006). Thus the theory of surplus-value is completed in the analysis of economic reproduction (Marx 1885, pt. 3), rather than by the presentation in Marx (1867).

5.3. Nonbasic Outputs and Surplus Product

> A man *grows rich* by employing a multitude of manufacturers: he *grows poor* by maintaining a multitude of menial servants. (Smith 1776, book II, ch. III; emphasis added)

This remark may merely seem to apply to an individual employer but in fact generalizes into a macroeconomic property.

Result 5.8 (Dependence on surplus labor): Production of nonbasic outputs is predicated on the extraction of surplus labor. More formally, if the share of surplus labor is $\sigma=0$ then the production of nonbasic outputs equals 0.[27]

Production of luxuries and other nonbasic outputs drains the surplus in the basic sector. Activities involved in such production impede the expansion of the basic sectors and are "unproductive" in the sense of classical political economy. In modern capitalist economies, this includes the arms industry and finance sector. Conversely, many socialized goods and services, such as public health care and education, are basic outputs and thus "productive."

Result 5.9 (Drain on the basic sectors): The surplus of basic outputs is impeded by the production of nonbasic outputs. More formally, let $\mathbf{b}$ and $\mathbf{b}'$ denote the net production of basic and nonbasic outputs, respectively, so that $\mathbf{n} = \mathbf{b} + \mathbf{b}'$. By redeploying labor from nonbasic to basic sectors, the surplus product in the latter sectors can be increased by

$$\mathbf{R}\mathbf{b}' + \frac{\mathbf{v}^*\mathbf{b}'}{\mathbf{v}^*\mathbf{b}}(\mathbf{I}-\mathbf{R})\mathbf{b} \geq \mathbf{0} \tag{17}$$

where the first term is the workers' consumption freed up from the nonbasic sectors and the second term is due to the increased production in the basic sectors.[28] Thus (17) represents a drain on the surplus capacity of the basic sectors incurred through the production of nonbasic outputs.[29]

Example 5.10 (Redeployment to basic sectors): Consider extending the net product in Example 2.5 with 50 units of sugar, which constitutes the nonbasic outputs. That is,

$$\mathbf{n} = \mathbf{b} + \mathbf{b}' = \begin{bmatrix} 10 \\ 100 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 50 \end{bmatrix}$$

Then the surplus product equals

$$\text{s} = \mathbf{n} - \mathbf{R}\mathbf{n} = \begin{bmatrix} 10 \\ 100 \\ 50 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 0.60 & 0.33 & 0.51 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 10 \\ 100 \\ 50 \end{bmatrix} = \begin{bmatrix} 10.00 \\ 35.50 \\ 50.00 \end{bmatrix}$$

and the share of surplus-value is $\sigma = 67\%$.

Suppose the total labor devoted to sustain the nonbasic sugar is redeployed to expand net output of basic iron and corn uniformly. Using (17), the form of the surplus product then changes into

$$\mathbf{s}' = \begin{bmatrix} 16.63 \\ 102.03 \\ 0 \end{bmatrix}$$

The share of surplus-value is still $\sigma = 67\%$ but the change has increased surplus iron and corn by +66% and +185%, respectively.

6. Conclusion

The classical economists assumed social labor as the basis of value on the ground that it was the original social cost of all produced commodities (see Ricardo 1817). An early attempt to instead deduce this relation can be found in Marx (1867, ch. 1) and was based on the observation that labor is a universal input that enters directly or indirectly into the production of every output-type. This line of reasoning was, however, necessarily incomplete since there are several other inputs that are universal in the same sense (Sraffa 1960).

Starting from an analysis of a reproducing economic system with a workforce, pioneered in Marx (1885, part III), we have shown that economic value can be derived as an endogenous property using the basic division of output given the consumption rates of the workforce. Specifically, we showed there is a unique valuation for which workers' consumption rates determine the distribution of value in a definitive proportion for any net product. The derivation reproduces the classical conception of value as direct and indirect labor requirements. Contrary to value derived in Sraffian theory, no additional assumptions about rates of return need to be made for surplus-producing economies. Moreover, we showed that Sraffian natural prices can be derived under a regime of constrained investments.

We discussed the possibility of deriving value in non-capitalist economies. We concluded the derivation is applicable to all economic systems with institutions capable of training and redeploying its finite amount of available labor time across different production processes. This, however, excludes for instance systems with rigid forms of caste hierarchies within the workforce. Our results, therefore, extend the applicability of classical economic concepts to the analysis of distribution, productivity, and employment in a wide range of economic systems.

Notes

1. Steele (1981) provides a review of criticisms against the idea that economic value could exist outside markets. In the analysis of the Russian Marxist economist Rubin,

> We could define abstract labour approximately as follows: Abstract labour is the designation for that part of the total social labour which was equalised in the process of social division of labour through the equation of *the products of labour on the market.* (Rubin 1978; emphasis added)

This line of thought has been advanced by the so-called value-form school (Heinrich 2012). An argument for why labor-based value only belongs to machine-based capitalist economies can be found in Onishi (2019).

2. Which Marx called "use values" (Marx 1867).
3. This is systematized in the international system of bar codes which associates a 12-digit number with each product kind.
4. The scalar must be dimensionless since each element of the vector is of a different algebraic type. The element b_{iron} would be measured in tons of iron and is thus incommensurate with element b_{corn} measured in units of bushels of corn, or b_{Ka} in units of single Ford Ka's. Dimensional analysis enables "sanity checks" of algebraic formulae, which can be economically meaningless otherwise (see Brody 1970; Fröhlich 2010; Valle Baeza 2010).
5. The ordering satisfies i) reflexivity: $\mathbf{b} \leq_{\mathrm{v}} \mathbf{b}$ and ii) transitivity: $\mathbf{b} \leq_{\mathrm{v}} \mathbf{b}'$ and $\mathbf{b}' \leq_{\mathrm{v}} \mathbf{b}''$ imply $\mathbf{b} \leq_{\mathrm{v}} \mathbf{b}''$. This in turn induces an equivalence relation $\mathbf{b} \sim_{\mathrm{v}} \mathbf{b}'$, for all pairs $(\mathbf{b},\mathbf{b}')$ that satisfy $\mathbf{b} \leq_{\mathrm{v}} \mathbf{b}'$ and $\mathbf{b}' \leq_{\mathrm{v}} \mathbf{b}$.
6. The consumption rate vector $\boldsymbol{\kappa}$ can be estimated from national accounts data using the inputs to the household sector and the total wage bill.
7. For sake of generality, we consider m units of production that may produce more than one output-type each. The necessary consumption by the workforce is $\boldsymbol{\kappa}$ multiplied by the total amount of labor employed in production, $\boldsymbol{l}\mathbf{x}$, where $\mathbf{x}$ is the $m\times 1$ vector of activity levels for all units and $\boldsymbol{l}$ is a $1\times m$ vector of labor input coefficients. The net product can be expressed as $\mathbf{n} = \mathbf{q} - \mathbf{A}\mathbf{x}$, where $\mathbf{q}$ is the $d\times 1$ vector of gross outputs in the economy and $\mathbf{A}$ is the $d\times m$ nonnegative matrix of input requirement coefficients. The activity levels are given by $\mathbf{x} = \mathbf{P}\mathbf{q}$, where $\mathbf{P}$ is a $m \times d$ nonnegative matrix of production allocation coefficients with columns that sum to unity. A necessary condition for economic reproduction is then $\lambda_{\max}(\mathbf{AP}) < 1$. The two equations can be stated as

$$\begin{bmatrix} \mathbf{n} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_d & -\mathbf{A} \\ -\mathbf{P} & \mathbf{I}_m \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{x} \end{bmatrix} \tag{1}$$

Then the triplet $(\boldsymbol{l}, \mathbf{A}, \mathbf{P})$ describes the technical conditions of production and can be estimated using data from supply and use tables (Lenzen and Rueda-Cantuche 2012; Eurostat 2008). With these relations in place, the necessary consumption can be expressed as

$$\boldsymbol{\kappa}(\boldsymbol{l}\mathbf{x}) = \boldsymbol{\kappa}\begin{bmatrix} \mathbf{0} & \boldsymbol{l} \end{bmatrix} \begin{bmatrix} \mathbf{q} \\ \mathbf{x} \end{bmatrix} = \boldsymbol{\kappa}\begin{bmatrix} \mathbf{0} & \boldsymbol{l} \end{bmatrix} \begin{bmatrix} \mathbf{I}_d & -\mathbf{A} \\ -\mathbf{P} & \mathbf{I}_m \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{n} \\ \mathbf{0} \end{bmatrix} = \left(\boldsymbol{\kappa}\boldsymbol{l}\mathbf{P}(\mathbf{I}_d - \mathbf{AP})^{-1}\right)\mathbf{n} = \mathbf{Rn} \tag{2}$$

using (1). If each output-type can be produced separately, then $m = d$ and we can let $\mathbf{P} = \mathbf{I}$, so that $\mathbf{R} = \boldsymbol{\kappa}\boldsymbol{l}(\mathbf{I}_d - \mathbf{A})^{-1}$ (see Shaikh 2016, equation 6.1.18).
8. That is, the input requirement coefficients are

$$\boldsymbol{l} = \begin{bmatrix} 0.6 & 0.2 \end{bmatrix} \text{ and } \mathbf{A} = \begin{bmatrix} 0 & 0.20 \\ 0 & 0.02 \end{bmatrix}, \tag{3}$$

assuming $\mathbf{P} = \mathbf{I}$ for simplicity.

9. More specifically, σ ∈ [0,1). Note that Marx's unbounded "rate of surplus-value" $\frac{\mathbf{vn}-\mathbf{vRn}}{\mathbf{vRn}}=\frac{\sigma}{1-\sigma}\in[0,\infty)$ is a mere transformation of σ.
10. Begin by rearranging (6) as $[(1-\sigma)\mathbf{v}-\mathbf{vR}]\mathbf{n}=\mathbf{0}$. For the real distribution σ to be determined by **κ** directly, the valuation vector must satisfy $(1-\sigma)\mathbf{v}=\mathbf{vR}$. That is, **v** is an eigenvector to **R** and σ is given by the solution to

$$\det((1-\sigma)\mathbf{I}-\mathbf{R})=0$$

Let λ = 1 – σ, then using the matrix determinant lemma along with **R** in (2) yields

$$\left(1-\boldsymbol{l}\mathbf{P}(\mathbf{I}-\mathbf{AP})^{-1}\kappa\lambda^{-1}\right)\lambda^{d}=0$$

Since λ = 0 corresponds to a workforce that does not consume anything, it is economically meaningless and only

$$\lambda=1-\sigma=\boldsymbol{l}\mathbf{P}(\mathbf{I}-\mathbf{AP})^{-1}\boldsymbol{\kappa}>\mathbf{0}$$

is a meaningful solution.
11. The intermediate labour requirements for each output-type are given by

$$\boldsymbol{l}(k)=\boldsymbol{l}\mathbf{P}(\mathbf{AP})^{k-1}$$

Using the invariance condition (7) and (2), we have that

$$(1-\sigma)\mathbf{v}^{*}=\mathbf{v}^{*}\mathbf{R}\equiv(\mathbf{v}^{*}\boldsymbol{\kappa})\boldsymbol{l}\mathbf{P}(\mathbf{I}-\mathbf{AP})^{-1} \quad (9)$$

After dividing (9) by 1 – σ>0, the nontrivial solution $\mathbf{v}\propto\boldsymbol{l}\mathbf{P}(\mathbf{I}-\mathbf{AP})^{-1}$ is obtained and is invariant with respect to the consumption-rate vector κ. Using the series expansion $\sum_{k=0}^{\infty}(\mathbf{AP})^{k}=(\mathbf{I}-\mathbf{AP})^{-1}$ it follows that

$$\mathbf{v}=\sum_{k=0}^{\infty}\boldsymbol{l}(k)$$

in (8).
12. The derivation of **v*** from (6) is based on the decomposition of the net product, and is not interpreted by evaluating "inputs and outputs" in production (whether "simultaneous" or "sequential" evaluation); see section 9 in the review by Foley (2000).
13. That is, a standard choice of numeraire is $\mathbf{v}^{*}\mathbf{n}=L$, where L is the total units of labor required to reproduce the net product.
14. Using (1) we have that

$$\mathbf{n}=(\mathbf{I}-\mathbf{AP})\mathbf{q}$$

where **APq** are input requirements in production. Then

$$\mathbf{APq}=\left(\mathbf{AP}(\mathbf{I}-\mathbf{AP})^{-1}\right)\mathbf{n}=\mathbf{Hn}.$$

15. At u=0, we have that

$$(\mathbf{I}-\mathbf{R})\mathbf{n} = g\mathbf{H}\mathbf{n}$$

which yields the eigenequation

$$\frac{1}{g}\mathbf{n} = (\mathbf{I}-\mathbf{R})^{-1}\mathbf{H}\mathbf{n}$$

Using the Perron-Frobenious theorems (Kurz and Salvadori 1997, sec. A.3),

$$g_{\max} = 1/\lambda_{\max}\left((\mathbf{I}-\mathbf{R})^{-1}\mathbf{H}\right)$$

16. Insert (6) into (10) to obtain

$$\left[(1-u)\mathbf{v}-\mathbf{v}(\mathbf{R}+g\mathbf{H})\right]\mathbf{n} = 0$$

To determine u independently of **n**, **v** must solve the eigenequation

$$(1-u)\mathbf{v} = \mathbf{v}(\mathbf{R}+g\mathbf{H})$$

Using the Perron-Frobenious theorems, the valid solution $\tilde{\mathbf{v}}$ is the eigenvector corresponding to the maximum eigenvalue, so that

$$u = 1-\lambda_{\max}(\mathbf{R}+g\mathbf{H})$$

is a function of g. After rearranging the eigenequation and using (2), we obtain

$$(1-u)\tilde{\mathbf{v}} = \tilde{\mathbf{v}}\mathbf{R}\left(\mathbf{I}-\frac{g}{1-u}\mathbf{H}\right)^{-1} \equiv (\tilde{\mathbf{v}}\boldsymbol{\kappa})(\mathbf{I}-\pi\mathbf{H})^{-1} \Rightarrow \tilde{\mathbf{v}} = w\mathbf{v}^{*}(\mathbf{I}-\pi\mathbf{H})^{-1} \quad (12)$$

Where

$$\pi = \frac{g}{1-u(g)}$$

and w is proportionality constant. Since $\tilde{\mathbf{v}}$ is determined up to a unit of choice, (11) follows from (12).

17. For equivalence, we require that $\mathbf{v}^{*}(\mathbf{I}-\pi\mathbf{H})^{-1} = \lambda\mathbf{v}^{*}$ in (11), i.e., $\mathbf{v}^{*}\mathbf{H} \propto \mathbf{v}^{*}$. Using the definitions of **v*** and **H** we have $\mathbf{v}^{*}\mathbf{AP} \propto \boldsymbol{l}\mathbf{P}$, which means that the value of inputs ("capital") requirements is proportional to direct labor requirements across all output-types. When $g = 0$ the result is trivial.
18. Equation (12) can be expressed as

$$\tilde{\mathbf{v}} = \pi\tilde{\mathbf{v}}\mathbf{H} + w\mathbf{v}^{*} = (\pi\tilde{\mathbf{v}}\mathbf{AP} + w\boldsymbol{l}\mathbf{P})\mathbf{v}^{*}(\mathbf{I}-\mathbf{AP})^{-1} = (1+\pi)\tilde{\mathbf{v}}\mathbf{AP} + w\boldsymbol{l}\mathbf{P}$$

which is of course Sraffa's natural prices in single-product systems (**P=I**) with an average profit rate

$$\pi = \frac{g}{1-u(g)} = \frac{g}{\lambda_{\max}(\mathbf{R}+g\mathbf{H})}$$

(Sraffa 1960; Pasinetti 1979; Kurz and Salvadori 1997).

19. Classical labor values therefore differ radically from the concept of "value" developed by the so-called value-form school. In the latter conception, there can be no abstract labor measured in hours nor can it be measured before the act of market exchange (Heinrich 2012, 50, 55, 65).

20. See for instance, Farjoun and Machover (1983), Cockshott and Cottrell (1998), Zachariah (2006), Shaikh (2016).
21. Friedman and Baker (2009) give several examples of scheduling constraints on new gun mountings, and slip sizes affecting UK destroyer construction plans in World War II. Friedman (2015) gives the example of construction of the Admiral class capital ships being postponed due to there not being enough shipbuilding labor to build both them and destroyers in 1917. For large-scale shipbuilding programs, even in peace, similar forward planning of physical constraints has to be done by the state (Arena et al. 2005).
22. Marx (1867, ch. 1, sec. 1, §18) points out:

> The value of a commodity would therefore remain constant, if the labour time required for its production also remained constant. But the latter changes with every variation in the productiveness of labour. This productiveness is determined by various circumstances, amongst others, by the average amount of skill of the workmen, the state of science, and the degree of its practical application, the social organisation of production, the extent and capabilities of the means of production, and by physical conditions.

23. The total employment requirement for producing n_i units of output i is $L_i = v_i n_i$. Therefore the relative change of employment is given by the identity $\dot{L}_i / L_i = -\rho_i + g_i$, where $g_i = \dot{n}_i / n_i$. If the actual employment is fixed, then the left-hand side is 0 and correspondingly $g_i = \rho_i$.
24. More generally, we may define

$$\tilde{\mathbf{A}} = (\mathbf{A} + \boldsymbol{\kappa}\boldsymbol{l})\mathbf{P}$$

and classify output i as basic if

$$\boldsymbol{e}_i^{\mathrm{T}}\left(\tilde{\mathbf{A}}^1 + \tilde{\mathbf{A}}^2 + \ldots + \tilde{\mathbf{A}}^d\right) > 0$$

We are naturally assuming that all consumption goods require some amount of direct labor. The concept is a slight generalization of Sraffa's "basic goods" and includes the production of the workers' consumption bundle. Note that the outputs that are basic and nonbasic may change over time as the structure of the economy changes (see Cockshott and Zachariah 2006). In the following, we will assume $\mathbf{P} = \mathbf{I}$ for simplicity.
25. Here, the input coefficients are

$$\boldsymbol{l} = \begin{bmatrix} 0.6 & 0.2 & 0.3 \end{bmatrix}$$

and

$$\mathbf{A} = \begin{bmatrix} 0 & 0.20 & 0.30 \\ 0 & 0.02 & 0.10 \\ 0 & 0 & 0 \end{bmatrix}$$

The real-consumption of the workforce $\boldsymbol{\kappa}$ is corn as before. Then

$$\tilde{\mathbf{A}} = (\mathbf{A} + \boldsymbol{\kappa}\boldsymbol{l})\mathbf{P} = \begin{bmatrix} 0 & 0.20 & 0.30 \\ 0.60 & 0.22 & 0.40 \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

which is upper block-triangular. Therefore corn and iron are basic outputs, while sugar is a nonbasic output.

26. Using the inverse of the upper block triangular matrix ($\mathbf{I} - \mathbf{A}$), we have that

$$\begin{aligned}
\mathbf{v}^* &= \boldsymbol{l}(\mathbf{I}-\mathbf{A})^{-1} \\
&= [\boldsymbol{l}_b \quad \boldsymbol{l}_u]\begin{bmatrix} (\mathbf{I}-\mathbf{A}_b)^{-1} & (\mathbf{I}-\mathbf{A}_b)^{-1}\mathbf{A}_{bu}(\mathbf{I}-\mathbf{A}_u)^{-1} \\ \mathbf{0} & (\mathbf{I}-\mathbf{A}_u)^{-1} \end{bmatrix} \\
&= \left[\boldsymbol{l}_b(\mathbf{I}-\mathbf{A}_b)^{-1} \quad \boldsymbol{l}_b(\mathbf{I}-\mathbf{A}_b)^{-1}\mathbf{A}_{bu}(\mathbf{I}-\mathbf{A}_u)^{-1} + \boldsymbol{l}_u(\mathbf{I}-\mathbf{A}_u)^{-1}\right] \\
&= \left[\mathbf{v}_b^* \quad \mathbf{v}_u^*\right]
\end{aligned} \tag{15}$$

Then it follows that $\sigma = 1 - \boldsymbol{l}(\mathbf{I}-\mathbf{A})^{-1}\boldsymbol{\kappa} = 1 - \mathbf{v}_b^*\boldsymbol{\kappa}_b$. Note that we also have

$$\begin{aligned}
\mathbf{v}^*\mathbf{R} &= \boldsymbol{\kappa}\boldsymbol{l}(\mathbf{I}-\mathbf{A})^{-1} \\
&= \mathbf{v}^*\begin{bmatrix} \boldsymbol{\kappa}_b \\ \mathbf{0} \end{bmatrix}\left[\boldsymbol{l}_b(\mathbf{I}-\mathbf{A}_b)^{-1} \quad \boldsymbol{l}_b(\mathbf{I}-\mathbf{A}_b)^{-1}\mathbf{A}_{bu}(\mathbf{I}-\mathbf{A}_u)^{-1} + \boldsymbol{l}_u(\mathbf{I}-\mathbf{A}_u)^{-1}\right] \\
&= \mathbf{v}^*\begin{bmatrix} \mathbf{R}_b & \mathbf{R}_{bu} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \\
&= \left[\mathbf{v}_b^*\mathbf{R}_b \quad \mathbf{v}_b^*\mathbf{R}_{bu}\right]
\end{aligned} \tag{16}$$

Then the left eigenequation $\lambda\mathbf{v}^* = \mathbf{v}^*\mathbf{R}$ yields two equations: $\lambda\mathbf{v}_b^* = \mathbf{v}_b^*\mathbf{R}_b$ and $\lambda\mathbf{v}_u^* = \mathbf{v}_b^*\mathbf{R}_{bu}$, and consequently $\sigma = 1 - \lambda$ is determined by the basic outputs $\mathbf{R}_b$.

27. By definition, $\mathbf{s} = \mathbf{n} - \mathbf{R}\mathbf{n} = (\mathbf{I}-\mathbf{R})\mathbf{n}$. Thus $\mathbf{v}^*\mathbf{s} = \sigma\mathbf{v}^*\mathbf{n} = 0$, when σ=0. Since $\mathbf{v}^*>0$ and $s \geq 0$ it follows that s = 0. Using the partitioning of R in (16), it follows that the net production of nonbasic outputs is a surplus product, that is, $\mathbf{n}_u = s_u$. Using $\mathbf{q} = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{n}$, we have that the activity levels in the nonbasic sectors equal $\mathbf{q}_u = (\mathbf{I}-\mathbf{A}_u)^{-1}\mathbf{n}_u = (\mathbf{I}-\mathbf{A}_u)^{-1}\mathbf{s}_u = 0$.

28. Consider redeploying the resources devoted to support the nonbasic sector to the basic sector alone. Let the net product before and after the change be $\mathbf{n}$ and $\mathbf{n}'$, respectively, where total employment remains the same, i.e., $\mathbf{v}^*\mathbf{n}' = \mathbf{v}^*\mathbf{n}$. Suppose the redeployment is such that the net product in the basic sector is increased uniformly by a factor α, i.e.,

$$\mathbf{n}' = \begin{bmatrix} (1+\alpha)\mathbf{n}_b \\ 0 \end{bmatrix}$$

Then it follows that the factor is

$$\alpha = \frac{\mathbf{v}_u^*\mathbf{n}_u}{\mathbf{v}_b^*\mathbf{n}_b}$$

The resulting change in the surplus product of the economy is

$$\Delta = \mathbf{s}' - \mathbf{s} = (\mathbf{I}-\mathbf{R})\mathbf{n}' - (\mathbf{I}-\mathbf{R})\mathbf{n} = (\mathbf{I}-\mathbf{R})\begin{bmatrix} \alpha\mathbf{n}_b \\ -\mathbf{n}_u \end{bmatrix} = \begin{bmatrix} \alpha(1-\mathbf{R}_b)\mathbf{n}_b + \mathbf{R}_{bu}\mathbf{n}_u \\ -\mathbf{n}_u \end{bmatrix} \tag{18}$$

where the top rows correspond to the basic sectors.

29. An early step towards such an analysis can be found in Marx (1885, ch. 20).

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